MSO, groups and graphs

Pytheas

Institut de mathématiques de Marseille

June 2021

Groups and graphs

- G finitely generated group
- G context-free = word problem of G is a context free language
- treewidth(G) = treewidth(undirected-Cayley(G))
- ...

MSO

- relational structure $S = < U; R_1, \ldots, R_k >$
- MSO formulas on S: $\phi(u, v) \equiv uR^* v$

$$\forall X, (u \in X \wedge (\forall x, y(x \in X \wedge xRy \implies y \in X))) \implies v \in X$$

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- $H = (V, E)$ a graph (if labeled, $E = E_1 \cup \cdots \cup E_k$)
- $MSO_1(H)$ when $U = V$ and $v_1 R v_2 \iff (v_1, v_2) \in E$
- $MSO_2(H)$ when $U = V \cup E$ and $vRe \iff e = (v, v')$

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- **Example:** K countably infinite clique

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- **Example:** K countably infinite clique

Theorem (Courcelle-1994)

If H is undirected and of bounded degree then $MSO_1(H)$ decidable **IFF** $MSO_2(H)$ decidable.

Decidability of MSO on f.g. groups

Corollary 4.1 (Kuske-Lohrey-2005)

If G is a finitely generated group, the following are equivalent:

- $MSO_1(Cayley(G))$ is decidable
- undirected-Cayley(G) has finite treewidth
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- Muller-Schupp-1983*: context-free = virtually free

Decidability of MSO on f.g. groups

Theorem 3.10 (Kuske-Lohrey-2005)

Let G be a labeled connected graph of bounded degree such that $Aut(G)$ has only finitely many orbits on G, then the following are equivalent:

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- (1) $\implies$ (2) : **Seese-1991** + Courcelle-1994
- (3) $\implies$ (1) : Muller-Schupp-1983 (see also Letichevskii-Smikun-1976?)

Theorem of Seese

- the hard one

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For any family of **planar** graphs for which MSO_1 is decidable, there is an uniform bound on the treewidth of graphs of the family.

Remember the clique... $\rightarrow$ **Seese's conjecture**

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Theorem (Seese-1976, Phd thesis?)

For any family of graphs for which MSO_2 is decidable, there is an uniform bound on the treewidth of graphs of the family.

proof: *sprichst du Deutsch?*

MSO engineering

Minors in MSO_2

- $R(x, y, D_V, D_E, C) \equiv$
  after *deleting vertices* in D_V and *deleting edges* in D_E and *contracting edges* in C, there is an edge from x to y

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- Remark: we can express in MSO_1 that H is a minor of G for a **fixed** $H \rightarrow$ planarity

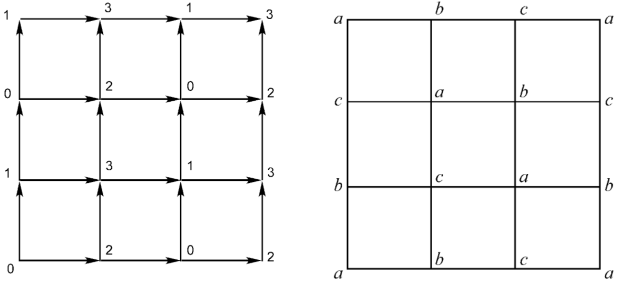
MSO engineering

$n \times n$-grids

MSO engineering

$n \times n$-grids

Proposition 5.14 page 340 of Courcelle-Engelfriet



In MSO_1

MSO engineering

Seese Theorem

MSO engineering

Seese Theorem

- with MSO, we can encode computation of a Turing machine on the grid starting from the SW-corner

- $M \mapsto \phi_M$

  $\phi_M \equiv$ *for any grid minor, the computation of* M *on this grid doesn't halt*

MSO engineering

Seese Theorem

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- $M \mapsto \phi_M$

  $\phi_M \equiv$ *for any grid minor, the computation of M on this grid doesn't halt*

- by the grid minor theorem, if G has infinite treewidth then G contains arbitrarily large grid minors and therefore

$$G \models \phi_M \iff M \text{ doesn't halt.}$$

- MSO_2 cannot be decidable on such G.